JOURNAL OF COMPUTATIONAL PHYSICS 24, 274–302 (1977)

# TOMCAT — A Code for Numerical Generation of Boundary-Fitted Curvilinear Coordinate Systems on Fields Containing Any Number of Arbitrary Two-Dimensional Bodies*

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Received March 19, 1976; revised August 12, 1976

The procedure and availability of the FORTRAN code for a method for automatic numerical generation of curvilinear coordinate systems with coordinate lines coincident with all boundaries of a general multiconnected, two-dimensional region containing any number of arbitrarily shaped bodies is described. No restrictions are placed on the shape of the boundaries, which may even be time dependent, and the approach is not restricted in principle to two dimensions. With this procedure the numerical solution of a partial differential system may be done on a fixed rectangular field with a square mesh with no interpolation required regardless of the shape of the physical boundaries, regardless of the spacing of the curvilinear coordinate lines in the physical field, and regardless of the movement of the coordinate system in the physical plane. A number of examples of coordinate systems and application thereof to the solution of partial differential equations are cited.

## 1. INTRODUCTION

There arises in all fields concerned with the numerical solution of partial differential equations the need for accurate numerical representation of boundary conditions. Such representation is best accomplished when the boundary is such that it is coincident with some coordinate line, for then the boundary can be made to pass through the points of a finite difference grid constructed on the coordinate lines; hence the choice of cylindrical coordinates for circular boundaries, elliptic coordinates for elliptical boundaries, etc. Finite difference expressions at, and adjacent to, the boundary may then be applied using only grid points on the intersections of coordinate lines, without the need for any interpolation between points of the grid.

The avoidance of interpolation is particularly important for boundaries with strong curvature or slope discontinuities, both of which are common in physical applications.

* Research Sponsored by NASA Langley Research Center, Grant NGR 25-001-055.

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ISSN 0021-9991

Likewise, interpolation between grid points not coincident with the boundaries is particularly inaccurate with differential systems that produce large gradients in the vicinity of the boundaries, and the character of the solution may be significantly altered in such cases. In most partial differential systems the boundary conditions are the dominant influence on the character of the solution, and the use of grid points not coincident with the boundaries thus places the most inaccurate difference representation in precisely the region of greatest sensitivity. The generation of a curvilinear coordinate system with coordinate lines coincident with all boundaries (herein called a "boundary-fitted coordinate system" for purposes of identification) is thus an essential part of a general numerical solution of a partial differential system.

A general method of generating boundary-fitted coordinate systems is to let the curvilinear coordinates be solutions of an elliptic partial differential system in the physical plane, with Dirichlet boundary conditions on all boundaries. One coordinate is specified to be constant on each of the boundaries, and a monotonic variation of the other coordinate around each boundary is specified. Thus, there is a coordinate line coincident with each boundary. The procedure, not restructed in principle to two dimensions, allows the coordinate lines to be concentrated as desired, and is applicable to all multiconnected regions (and thus to fields containing any number of arbitrarily shaped bodies). This automatic numerical generation of boundary-fitted coordinate systems has been discussed by the authors in Refs. [1, 2].

This general idea has been applied previously to two-dimensional regions interior to a closed boundary (simply-connected regions) by Winslow [3], Barfield [4], Chu [5], Amsden and Hirt [6], and Godunov and Prokopov [7]. Winslow [3] and Chu [5] took the transformed coordinates to be solutions of Laplace's equation in the physical plane which, as is shown in the next section, makes the physical cartesian coordinates solutions of a quasi-linear elliptic system in the transformed plane. Barfield [4] and Amsden and Hirt [6] reversed the procedure, taking the physical coordinates to be solutions in the transformed plane of a linear elliptic system which consists of Laplace's equation modified by a multiplicative constant on one term. This makes the transformed coordinates solutions of a quasi-linear elliptic system in the physical plane. Barfield also considered a hyperbolic system, but such a system cannot be used to treat general closed boundaries, since only elliptic systems allow specification of boundary conditions on the entirety of closed boundaries. Stadius [8] also used a hyperbolic system to generate a coordinate system for a doubly-connected region having parallel inner and outer boundaries. With parallel boundaries it is only necessary to specify conditions on one of the boundaries, the location of the other boundary being free. The elliptic system, however, allows all bonudaries to be specified as desired and thus has much greater flexibility.

Amsden and Hirt [6] constructed the coordinate generation method by iterative weighted averaging of the values of the physical coordinates at fixed points in the transformed plane in terms of values at neighboring points. Although not stated as such, this procedure is precisely equivalent to solving Laplace's equation, or modification thereof of the form noted above in Barfield [4], for the physical coordinates in the transformed plane by Gauss–Seidel iteration. Amsden and Hirt also allowed the boundary to move at each iteration, but this is simply equivalent to approaching the solution of the boundary-value problem through a succession of boundary-value problems converging to the problem of interest. In the approach of Godunov and Prokopov [7] the elliptic system is quasi-linear in both the physical and transformed planes. These authors applied a second transformation to that used by Chu [5], the transformation functions of this latter transformation being chosen a priori to control the coordinate spacing. Though not stated as such, the overall transformation may be shown to be generated by taking the transformed coordinats to be solutions in the physical plane of Laplace's equation modified by the addition of a multiple of the square of the Jacobian, the multiplicative factors being a priori chosen functions of the physical coordinates.

Meyder [9] generated an orthogonal curvilinear system by solving for the potential and "force" lines in a simply-connected region and taking these as the coordinate lines. This amounts to making the curvilinear coordinates solutions of Laplace equations in the physical plane with Dirichlet boundary conditions (constant) on part of the boundary and Neumann boundary conditions (vanishing normal derivative) on the remainder. The solution for the coordinates was done, however, in the physical plane on a rectangular grid using interpolation at the curved boundaries, rather than in the tansformed plane.

Orthogonal curvilinear coordinates for multiconnected regions, including regions with two bodies, have been generated by Ives [10] using conformal mapping. Conformal mapping is a special case of the generation of coordinate systems by solving an elliptic boundary-value problem, but is not extendable to three dimensions and is less flexible in the spacing of the coordinate lines.

In the present research, the technique of generating the transformed coordinates as solutions of an elliptic differential system in the physical plane has been applied to multiconnected regions with any number of arbitrarily shaped bodies (or holes). The elliptic equations for the coordinates are solved in finite difference approximation by SOR iteration. Procedures for controlling the coordinate system so that coordinate lines can be concentrated as desired have been developed. Initial effort was confined to two dimensions in the interest of computer economy, but the technique is extendable in principle to three dimensions. The procedure is also applicable to fields with time-dependent boundaries, one coordinate line remaining fixed to the moving boundary. Here the equations for the coordinates must be resolved at each time step. The computational grid remains fixed in spite of the movement of the physical grid.

Any partial differential system can be solved on the boundary-fitted coordinate system by transforming the set of partial differential equations of interest, and associated boundary conditions, to the curvilinear system. Since the boundary-fitted coordinate system has coordinate lines coincident with the surface contours of all bodies present, all boundary conditions can be expressed at grid points, and normal derivatives on the bodies can be represented using only finite differences between grid points on coordinate lines, without need of any interpolation, even though the coordinate system is not orthogonal at the boundary. The transformed equations can then be approximated using finite difference expressions and solved numerically in the transformed plane. Thus, regardless of the shape of the physical boundaries, and regardless of the spacing of the finite grid in the physical field, all computations, both to generate the coordinate system and, subsequently, to solve the partial differential system of interest can be done on a rectangular field with a square mesh with no interpolation required on the boundaries. Moreover, the physical boundaries may even be time dependent without affecting the grid in the transformed region.

The computer software utilized to generate the boundary-fitted coordinate system is independent of the set of partial differential equations to be solved on this system. For example, numerical solutions for inviscid and viscous fluid flows have been obtained using this system [11–15]. The partial differential equations governing these phenomena differ drastically. However, for a given body geometry, the same boundary-fitted system generation program was used in both solutions. Another major advantage of using boundary-fitted coordinates is that the computer software generated to approximate the solution of a given set of partial differential equations is completely independent of the physical geometry of the problem. The coordinate systems for the wide variety of bodies cited in this report, for example, were all developed utilizing the same computer program.

The computer code for generation of boundary-fitted coordinate systems for fields containing any number of arbitrarily shaped bodies is now available for general use [2], and its capabilities and application are discussed in the following sections. A complete discussion of the procedure, the complete computer code with instructions for its use, and several examples and test cases are included in Ref. [2]. The technique of the use of the boundary-fitted coordinate systems in the numerical solution of partial differential equations is illustrated in Ref. [13].

## 2. Coordinate System Generation

### A. *Mathematical Development*

The basic ideas of the generation of boundary-fitted curvilinear coordinate systems have been presented earlier [1], and will only be summarized here. Complete detail is given in Ref. [2].

Let it be desired to transform the two-dimensional multiconnected region $D$, bounded by three simple closed arbitrary contours, onto a rectangular region $D^*$. One such transformation for two bodies is illustrated in Fig. 1. The bodies are connected with one arbitrary cut, with an additional cut joining one of the body contours to the outer boundary. The physical plane contours $\Gamma_1 - \Gamma_8$ map, respectively, onto the contours $\Gamma_1{}^* - \Gamma_8{}^*$ in the transformed plane. Note that the body defined by the union of $\Gamma_7$ and $\Gamma_8$ is split into two segments ($\Gamma_7{}^*$ and $\Gamma_8{}^*$), as is the cut joining this body and the one defined by contour $\Gamma_1$. The $\eta$-coordinate is the same for both of the bodies and cut between them. Conversely, the cut defined by $\Gamma_3$ and $\Gamma_4$ in the physical plane is taken as a $\xi$ = constant line in the transformed plane. The outer boundary contour $\Gamma_2$ maps onto the upper boundary in the $[\xi, \eta]$ plane,

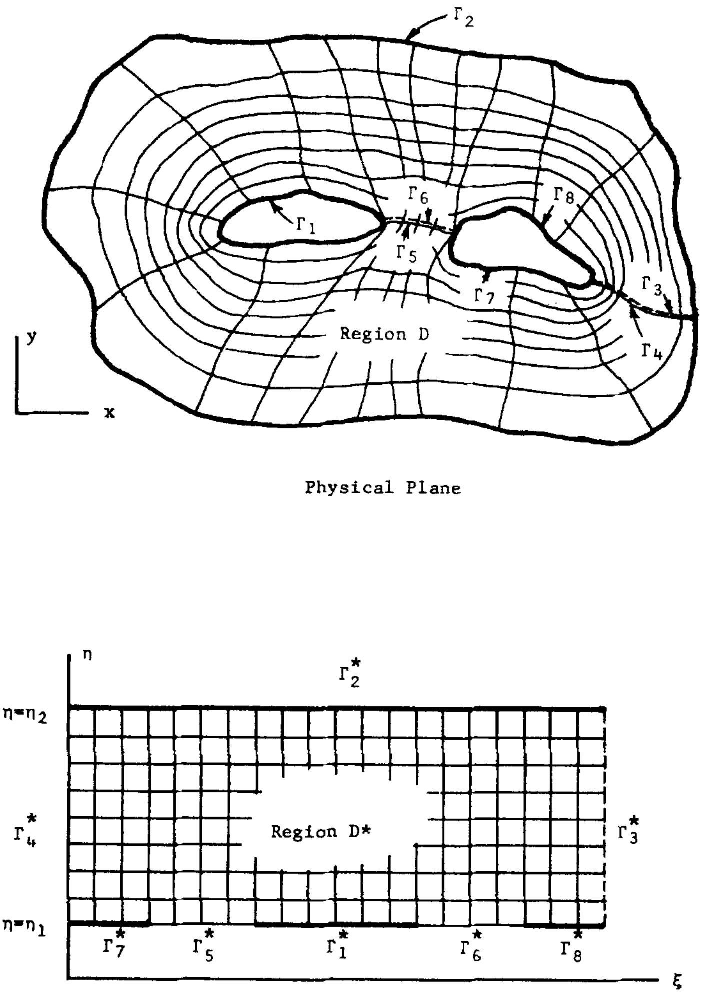



FIG. 1. Field transformation—Multiple bodies.

becoming a constant $\eta$-line. Two reentrant boundaries occur for this two-body transformation. The left and right vertical boundaries ($\Gamma_4{}^*$, $\Gamma_3{}^*$) are coincident in the physical plane and thus constitute reentrant boundaries in the transformed plane. In addition a horizontal reentrant segment due to the coincidence of $\Gamma_5$ and $\Gamma_6$ in the physical plane arises. The coordinate functions and the derivatives thereof are continuous across these reentrant boundaries.

Certain considerations must be taken into account in the choice of a suitable elliptic generating system for the coordinates as discussed in Ref. [2]. The system chosen allows considerable control of the coordinate line spacing as is illustrated in a later section. Control of the spacing of the coordinate lines on the body is easily accomplished, since the points on the body are input to the program. The spacing of the coordinate lines in the field, however, must be controlled by varying the elliptic generating system for the coordinates. One method of variation is to add inhomogeneous terms to the right sides of Laplace equations, so that the generating system becomes

$$\xi_{xx} + \xi_{yy} = P(\xi, \eta), \qquad \eta_{xx} + \eta_{yy} = Q(\xi, \eta) \tag{1}$$

with the Dirichlet boundary conditions

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi_1(x, y)\\ \eta_1\end{pmatrix}, \quad [x, y] \in \Gamma_1 \qquad \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi_2(x, y)\\ \eta_2\end{pmatrix}, \quad [x, y] \in \Gamma_2$$

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi_7(x, y)\\ \eta_1\end{pmatrix}, \quad [x, y] \in \Gamma_7 \qquad \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi_8(x, y)\\ \eta_1\end{pmatrix}, \quad [x, y] \in \Gamma_8$$

where $\eta_1$ and $\eta_2$ are different constants ($\eta_2 > \eta_1$), and $\xi_1(x, y)$, $\xi_2(x, y)$, $\xi_7(x, y)$, and $\xi_8(x, y)$ are specified monotonic functions on $\Gamma_1$, $\Gamma_2$, $\Gamma_7$, and $\Gamma_8$, respectively. The arbitrary curve joining $\Gamma_7$ with $\Gamma_2$, and $\Gamma_8$ with $\Gamma_2$, in the physical plane, which transforms to the left and right sides of the transformed plane, specifies a branch cut for the multiple-valued function $\xi(x, y)$. Thus, the values of the physical coordinate functions $x(\xi, \eta)$ and $y(\xi, \eta)$ are the same on $\Gamma_3$ as on $\Gamma_4$, and these functions and their derivatives are continuous from $\Gamma_3$ to $\Gamma_4$. Therefore, boundary conditions are neither required nor allowed on $\Gamma_3$ and $\Gamma_4$. The same comments apply to the other pair of reentrant boundaries, $\Gamma_5$ and $\Gamma_6$, which form a branch cut between the two bodies.

Since it is desired to perform all numerical computations in the uniform rectangular transformed plane, the dependent and independent variables must be interchanged in (1). In the transformed plane these equations become

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) = 0, \tag{2a}$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) = 0, \tag{2b}$$

where

$$\alpha \equiv x_\eta^2 + y_\eta^2, \qquad \gamma \equiv x_\xi^2 + y_\xi^2,$$

$$\beta \equiv x_\xi x_\eta + y_\xi y_\eta, \qquad J = x_\xi y_\eta - x_\eta y_\xi,$$

with the transformed boundary conditions

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}f_1(\xi, \eta_1)\\ f_2(\xi, \eta_1)\end{pmatrix}, \quad [\xi, \eta_1] \in \Gamma_1^* \qquad \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}g_1(\xi, \eta_2)\\ g_2(\xi, \eta_2)\end{pmatrix}, \quad [\xi, \eta_2] \in \Gamma_2^*$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}h_1(\xi, \eta_1)\\ h_2(\xi, \eta_1)\end{pmatrix}, \quad [\xi, \eta_1] \in \Gamma_7^* \qquad \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}q_1(\xi, \eta_1)\\ q_2(\xi, \eta_1)\end{pmatrix}, \quad [\xi, \eta_1] \in \Gamma_8^*,$$

The functions $f_1$, $f_2$, $g_1$, $g_2$, $h_1$, $h_2$, $q_1$, and $q_2$ are specified by the known shape of the contours $\Gamma_1$, $\Gamma_2$, $\Gamma_7$, and $\Gamma_8$ and the specified distribution of $\xi$ thereon. As noted, boundary data are neither required nor allowed along the reentrant boundaries $\Gamma_3^*$, $\Gamma_4^*$, $\Gamma_5^*$, and $\Gamma_6^*$.

The system given by the equations of (2) is a quasi-linear elliptic system for the physical coordinate functions, $x(\xi, \eta)$ and $y(\xi, \eta)$, in the transformed plane. Although this system is considerably more complex than that given by (1), the boundary conditions are specified on straight boundaries, and the coordinate spacing in the transformed plane is uniform. The boundary-fitted coordinate system generated by the solution to (2) has a constant $\eta$-line coincident with each boundary in the physical plane. The $\xi$ = constant lines may be spaced as desired around the boundaries, since the assignment of the $\xi$-values to the $[x, y]$ boundary points via the functions $f_1, f_2, g_1, g_2, h_1, h_2, q_1$, and $q_2$ is arbitrary. (Numerically, the discrete boundary values $[x_k, y_k]$ are transformed to equispaced discrete $\xi_k$-points on both boundaries.) Control of the radial spacing of the $\eta$ = constant lines and of the incidence angle of the $\xi$ = constant lines at the boundaries is accomplished by varying the functions $P(\xi, \eta)$ and $Q(\xi, \eta)$ in (2). All numerical computations, both to generate the boundary-fitted coordinate system and subsequently to utilize the coordinates for solving a set of partial differential equations, are executed on a rectangular field with a uniform grid.

The effect of changing the functions $P(\xi, \eta)$ and $Q(\xi, \eta)$ on the coordinate system is discussed in Ref. [2]. One particularly effective procedure is to choose $P$ and $Q$ as exponential terms, so that the coordinates are generated as the solutions of

$$\begin{aligned}\xi_{xx} + \xi_{yy} &= -\sum_{i=1}^{n} a_i \operatorname{sgn}(\xi - \xi_i) \exp(-c_i \mid \xi - \xi_i \mid)\\ &\quad - \sum_{j=1}^{m} b_j \operatorname{sgn}(\xi - \xi_j) \exp(-d_j((\xi - \xi_j)^2 + (\eta - \eta_j)^2)^{1/2})\\ &\equiv P(\xi, \eta),\end{aligned} \tag{3a}$$

$$\begin{aligned}\eta_{xx} + \eta_{yy} &= -\sum_{i=1}^{n} a_i \operatorname{sgn}(\eta - \eta_i) \exp(-c_i \mid \eta - \eta_i \mid)\\ &\quad - \sum_{j=1}^{m} b_j \operatorname{sgn}(\eta - \eta_j) \exp(-d_j((\xi - \xi_j)^2 + (\eta - \eta_j)^2)^{1/2})\\ &\equiv Q(\xi, \eta),\end{aligned} \tag{3b}$$

where the positive amplitudes and decay factors are not necessarily the same in the two equations. Here the first terms have the effect of attracting the $\xi$ = constant lines to the $\xi = \xi_i$ lines in Eq. (3a), and attracting $\eta$ = constant lines to the $\eta = \eta_i$ lines in Eq. (3b). The second terms cause $\xi$ = constant lines to be attracted to the points $(\xi_j, \eta_j)$ in (3a), with similar effect on $\eta$ = constant lines in (3b). Several examples of the use of coordinate system coltrol are given in Section 3.

## B. *Numerical Solution*

All derivatives in Eq. (2) are approximated by second-order central finite difference expressions. The resulting difference equations are given in Ref. [2]. The set of non-linear simultaneous difference equations is solved by point SOR iteration.

The actual values of the curvilinear coordinates, $\xi$ and $\eta$, are irrelevant to the subsequent use of the coordinate system in the numerical solution of partial differential equations, for the mesh widths in the transformed plane, $\Delta\xi$ and $\Delta\eta$, simply cancel out of all difference expressions for transformed derivatives. Therefore $\Delta\xi$ and $\Delta\eta$ are both taken as unity for convenience, with $\xi$ and $\eta$ each ranging from unity to the total number of coordinate lines of each description.

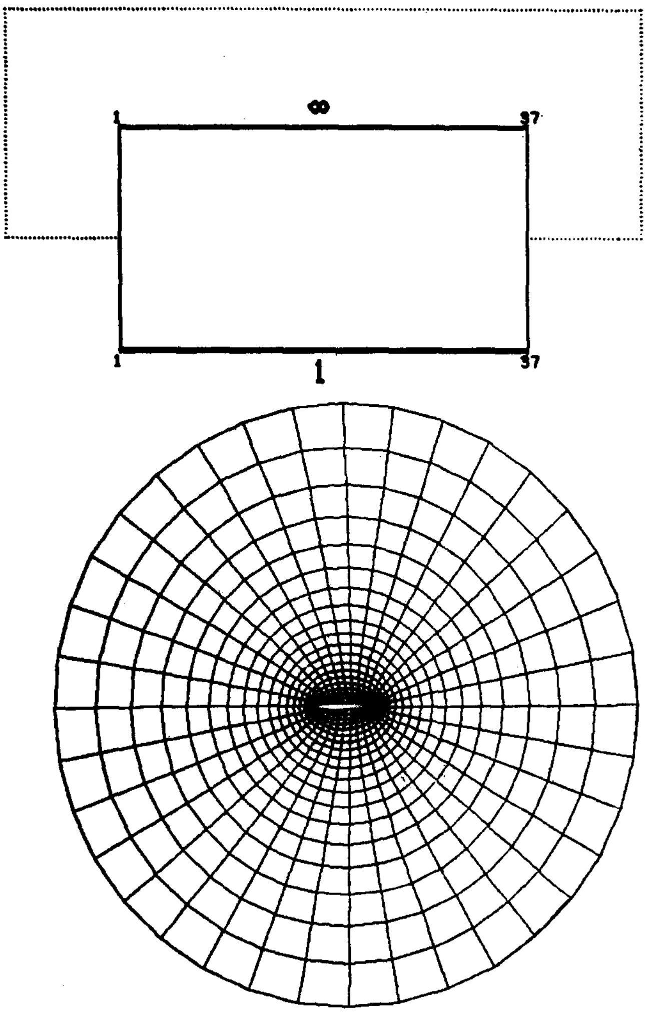



FIG. 2. Single-body configuration.

### C. *Multiple-Body Segment Arrangements*

In the case of a single body it is logical to keep the body contour in one segment, with a single cut connecting the single segment to the outer boundary. This type of arrangement is illustrated in Fig. 2. (Figure 3 shows an alternate single body arrangement. In these and all subsequent figures the dotted lines on the segment arrangement diagrams identify the two members of a reentrant pair.) In the case of multiple bodies there is a wider choice of resonable arrangements, some of which may be better than others for certain applications. The boundaries in the physical plane may be split into as many segments as desired, and these segments may be arranged around the rectangular boundary of the transformed plane in any way desired. These segments are all connected by branch cuts in the physical plane and by reentrant boundaries in the transformed plane. Several of these arrangements are illustrated in Figs. 4–12. Illustrative values of the segment input parameters are given in Ref. [2] for each of these arrangements.

In the arrangement of Fig. 4, an $\eta$-line encircles both bodies and forms a cut between the bodies, the cut to the outer boundary being a $\xi$-line. The outer boundary

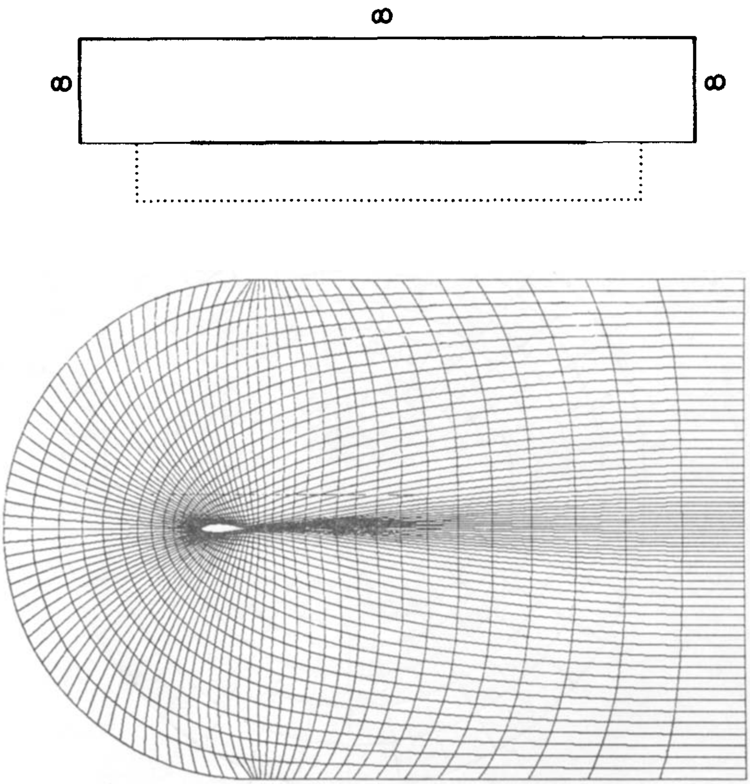



FIG. 3. Alternate single-body configuration.

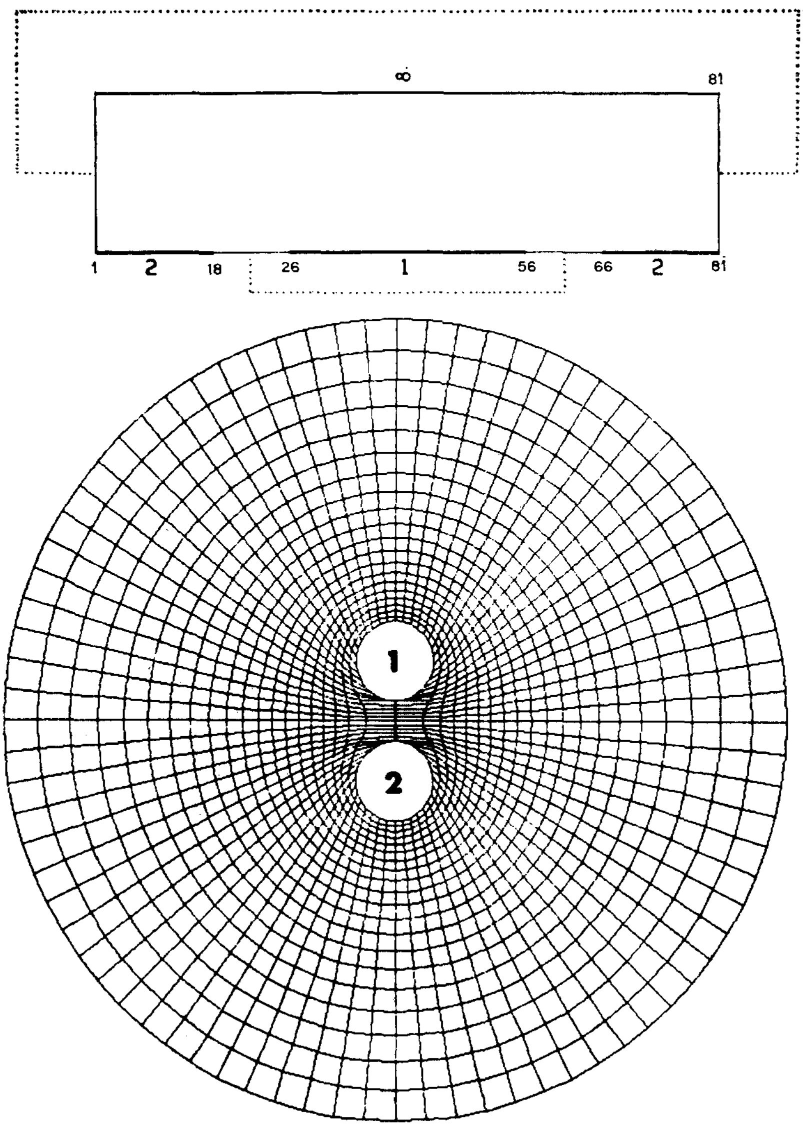
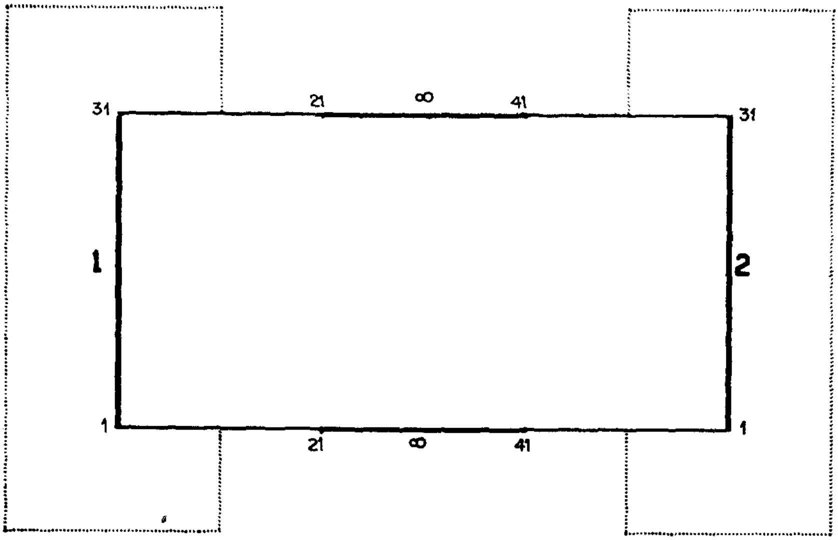



FIG. 4. Double-body segment configuration No. 1.

is also a line of constant $\eta$, but at a different value. Note that one body is split into two segments, while the other body and the outer boundary are each in single segments. Figure 5 shows an arrangement in which each body is in a single segment, each body being a $\xi$-line of different value. Here there is no cut between the bodies, but rather an $\eta$-line cut between each body and the outer boundary, which is split into two segments, each being an $\eta$-line of different value. (This produces a system similar



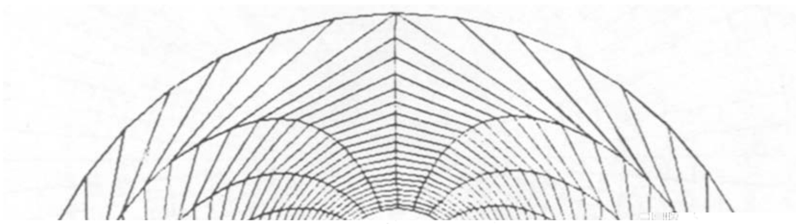



FIG. 5. Double-body segment configuration No. 2.

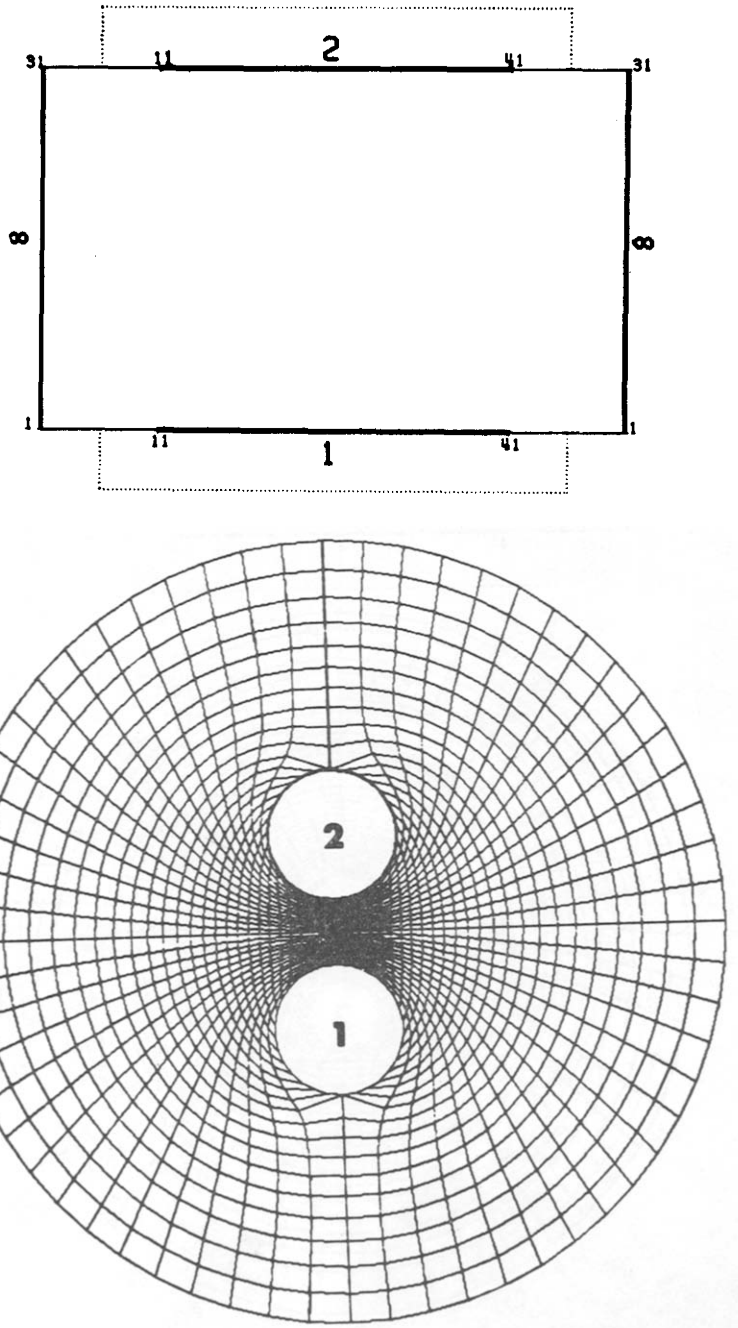
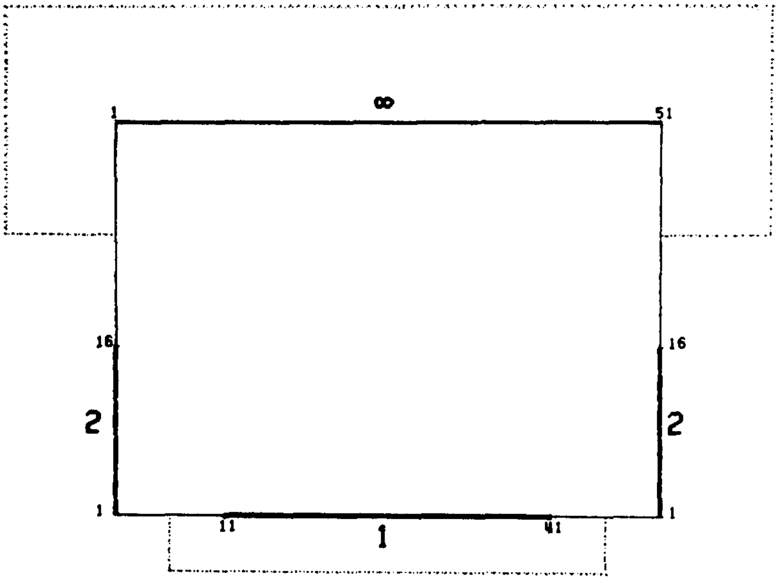



FIG. 6. Double-body segment configuration No. 3.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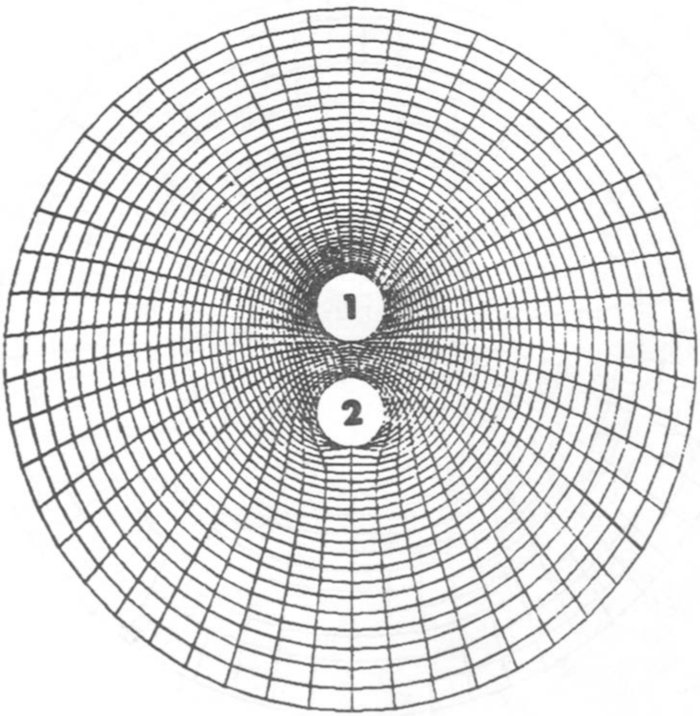
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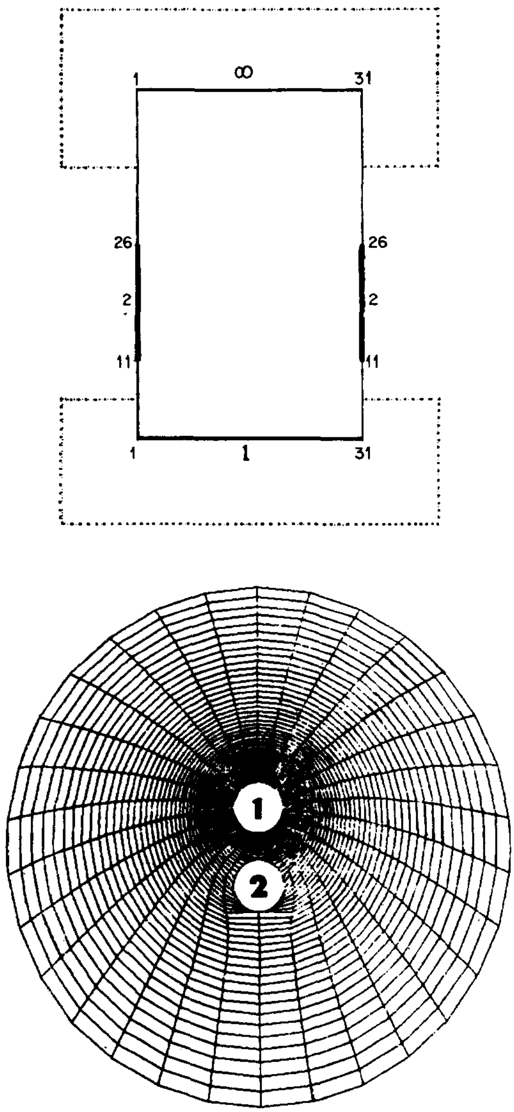


FIG. 8. Double-body segment configuration No. 5.

to a bipolar coordinate system.) In Fig. 6 each body is also in a single segment with the outer boundary split into two segments, but here an $\eta$-line encircles each body and forms the cut between that body and the outer boundary. In Fig. 7 one body is a single segment encircled by an $\eta$-line which forms a cut to the other body. The other body is split into two segments, each being a $\xi$-line of different value, with each segment connected to the outer boundary by a $\xi$-line. The outer boundary is in a

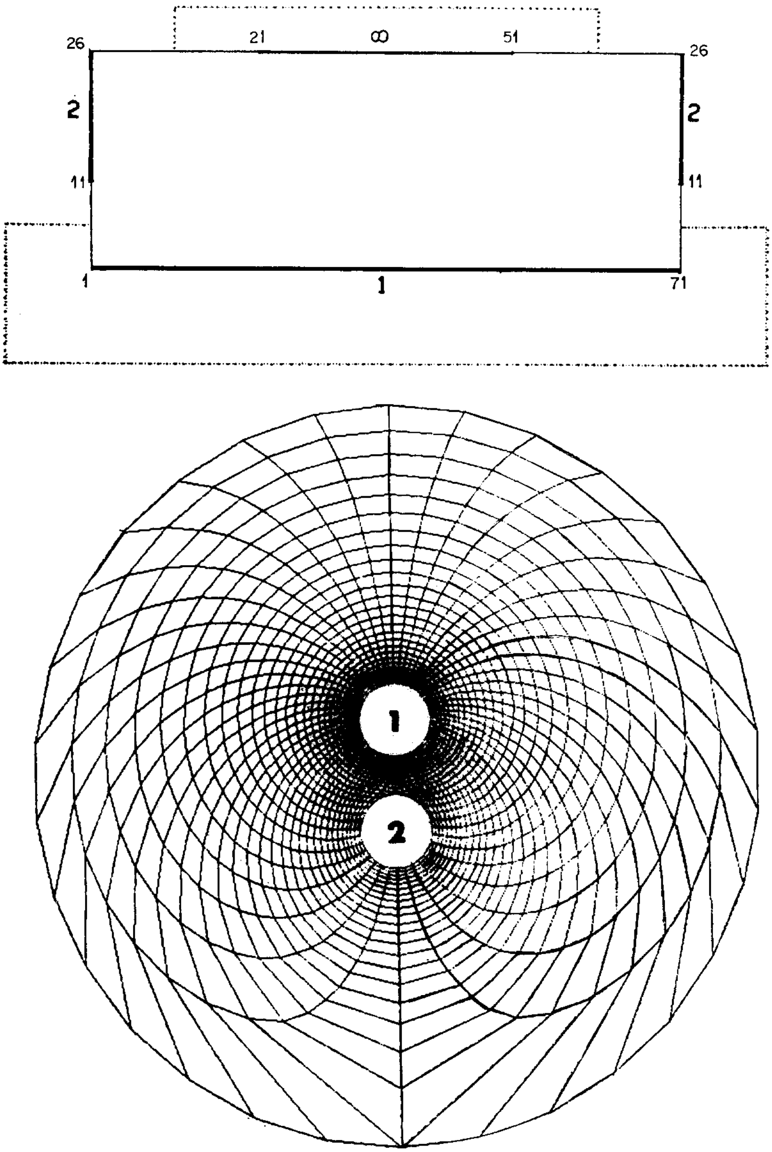


FIG. 9. Double-body segment configuration No. 6.

single segment and is an $\eta$-line. Other arrangements are shown in Figs. 8–12. All these arrangements are shown without coordinate line attraction, and, consequently, many of the resulting systems exhibit wide spacing in concave areas. This spacing can be improved by coordinate attraction as illustrated in the examples given in Section 3.

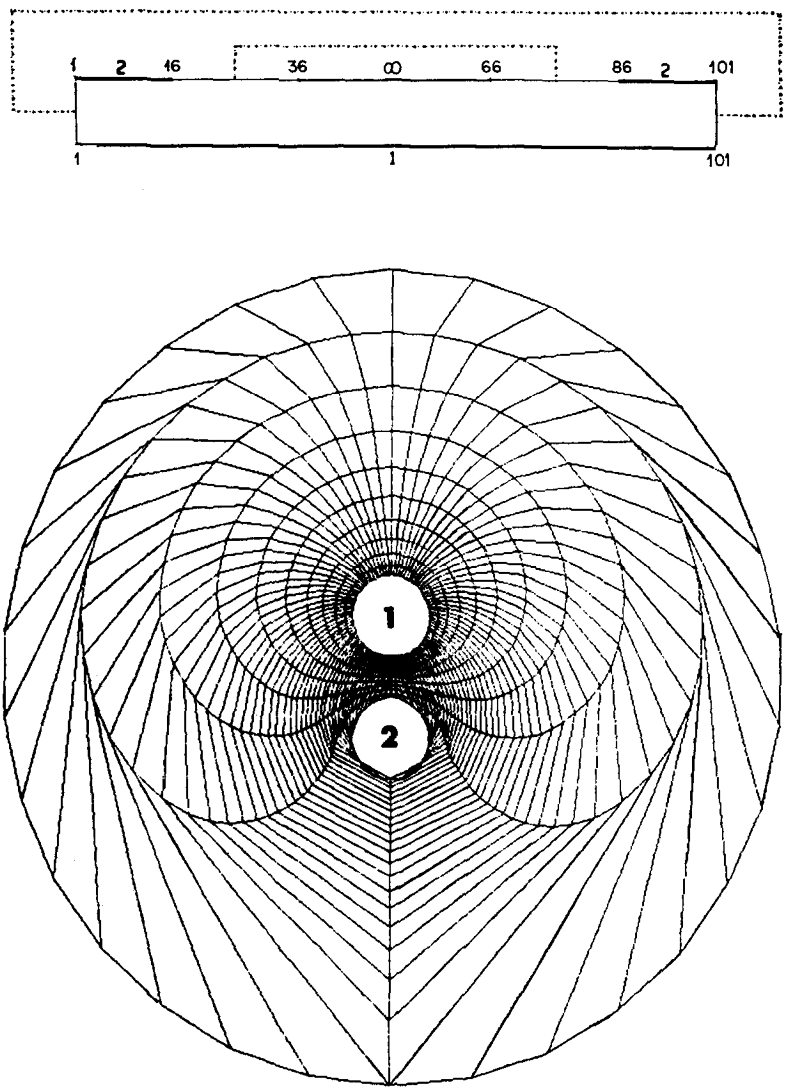


FIG. 10. Double-body segment configuration No. 7.

D. *Convergence Acceleration*

For a difference equation of the general form

$$a_1(f_{i+1,j} + f_{i-1,j}) + a_2(f_{i,j+1} + f_{i,j-1}) + b_1(f_{i+1,j} - f_{i-1,j}) \\ + b_2(f_{i,j+1} - f_{i,j-1}) + cf_{ij} + d_{ij} = 0 \qquad (i = 1, 2,..., I; j = 1, 2,..., J) \quad (4)$$

with boundary values specified on $i = j = 0$, $i = I + 1$, and $j = J + 1$, and $a_1$, $a_2$, $b_1$, $b_2$, $c$, and $d$ constant, the optimum value of the SOR acceleration parameter $\omega$

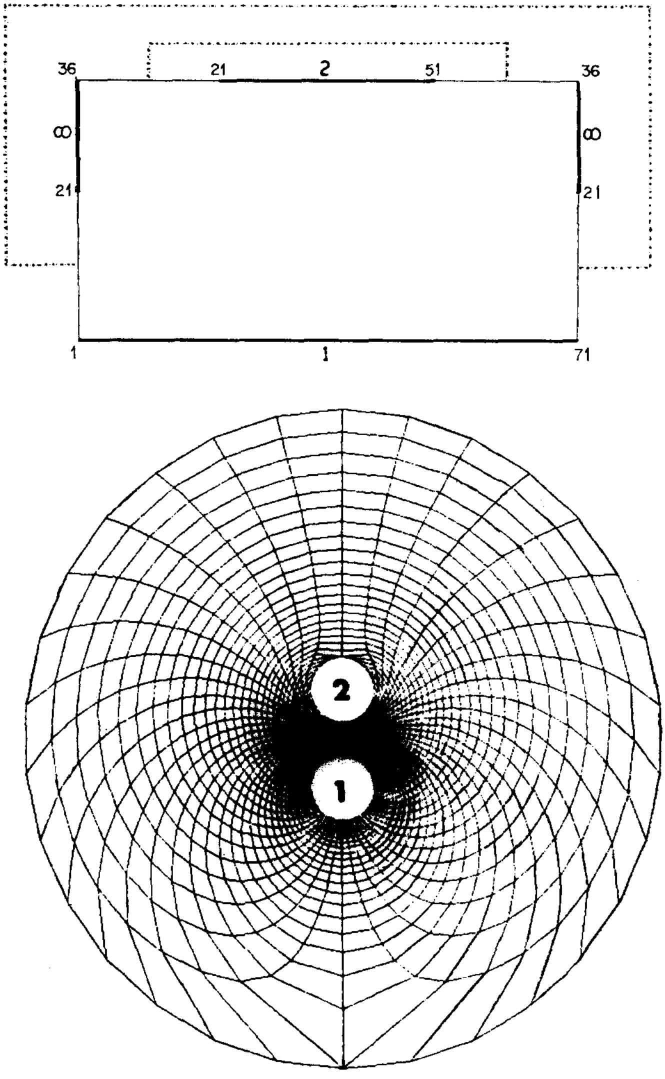


FIG. 11. Double-body segment configuration No. 8.

can be obtained in the case where $a_1^2 \geqslant b_1^2$ and $a_2^2 \geqslant b_2^2$, and in the case where $a_1^2 \leqslant b_1^2$ and $a_2^2 \leqslant b_2^2$, using the techniques of Ref. [16]. The optimum parameters in these two cases are

*Case* 1. $a_1^2 \geqslant b_1^2$ and $a_2^2 \geqslant b_2^2$

$$\omega = \frac{2}{1 + (1 - \rho^2)^{1/2}} \qquad \text{(overrelaxation, } 1 \leqslant \omega < 2\text{).} \tag{5}$$

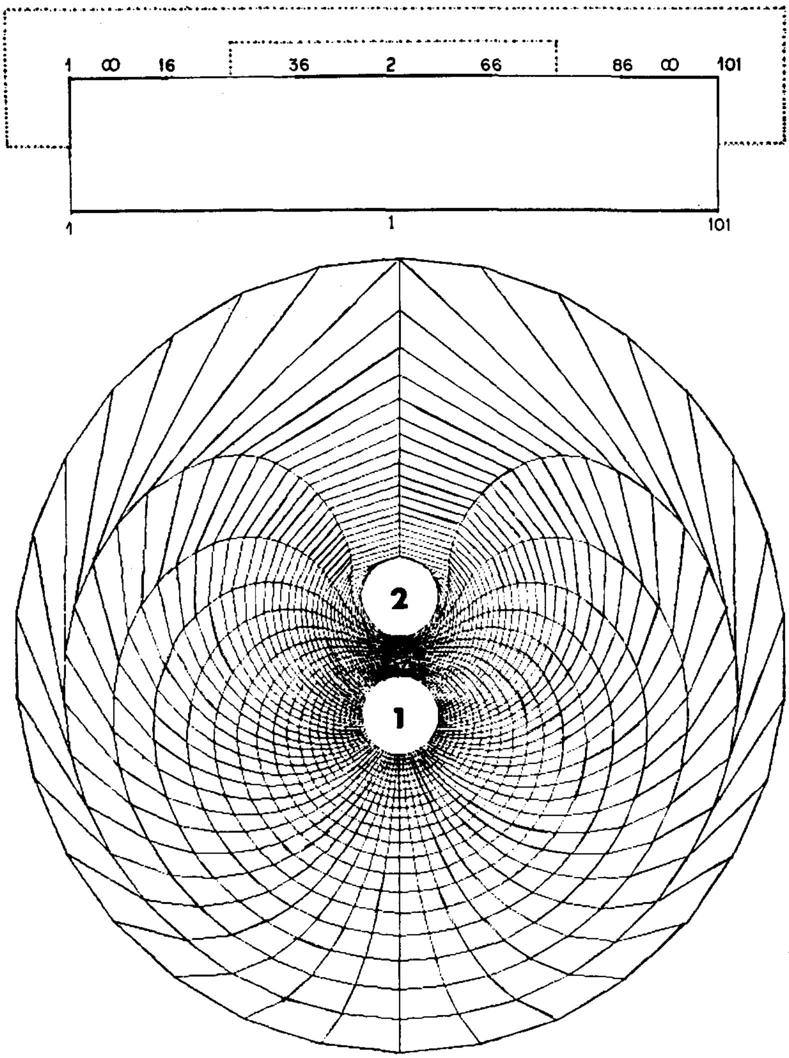


FIG. 12. Double-body segment configuration No. 9.

*Case* 2. $a_1^2 \leqslant b_1^2$ and $a_2^2 \leqslant b_2^2$

$$\omega = \frac{2}{1 + (1 + \rho^2)^{1/2}} \qquad \text{(underrelaxation, } 0 < \omega \leqslant 1) \tag{6}$$

where

$$\rho = 2((a_1/c)^2 - (b_1/c)^2)^{1/2} \cos(\pi/(I + 1)) + 2((a_2/c)^2 - (b_2/c)^2)^{1/2} \cos(\pi/(J + 1)). \tag{7}$$

In the remaining case where $a_1^2 \gtrless b_1^2$ and $a_2^2 \lessgtr b_2^2$, no theoretical determination of the optimum acceleration parameter exists as yet.

Since the difference equations for the coordinate system are nonlinear, the above theory is not directly applicable. However, if the equations are considered as locally linearized, then a local optimum acceleration parameter can be obtained which will vary over the field. It should be noted that the local linearization is applied only to the determination of the acceleration parameters, not the actual solution of the difference equations.

Following this approach and neglecting the effect of the cross derivatives, the local constants in the above equations as applied to (2) become

$$a_1 = \alpha, \qquad a_2 = \gamma,$$
$$b_1 = J^2P/2, \qquad b_2 = J^2Q/2,$$
$$c = -2(\alpha + \gamma),$$

so that the locally optimum acceleration parameters are, for an IMAX · JMAX field

*Case* 1. $\alpha_{ij} \geqslant J_{ij}^2 \mid P_{ij} \mid/2$ and $\gamma_{ij} \geqslant J_{ij}^2 \mid Q_{ij} \mid/2$

$$\omega_{ij} = \frac{2}{1 + (1 - \rho_{ij}^2)^{1/2}} \qquad \text{(overrelaxation).} \tag{8}$$

*Case* 2. $\alpha_{ij} \leqslant J_{ij}^2 \mid P_{ij} \mid/2$ and $\gamma_{ij} \leqslant J_{ij}^2 \mid Q_{ij} \mid/2$

$$\omega_{ij} = \frac{2}{1 + (1 + \rho_{ij}^2)^{1/2}} \qquad \text{(underrelaxation)} \tag{9}$$

where

$$\rho_{ij} = \frac{1}{\alpha_{ij} + \gamma_{ij}} \left[ \left( \alpha_{ij}^2 - \frac{J_{ij}^4 P_{ij}^2}{4} \right)^{1/2} \cos\left( \frac{\pi}{\text{IMAX} - 1} \right) + \left( \gamma_{ij}^2 - \frac{J_{ij}^4 Q_{ij}^2}{4} \right)^{1/2} \cos\left( \frac{\pi}{\text{JMAX} - 1} \right) \right]. \tag{10}$$

In the remaining case where $\alpha \gtrless J^2 \mid P \mid/2$ and $\gamma \lessgtr J^2 \mid Q \mid/2$, not even a local optimum is available. The program allows a choice of strategy in this case: overrelaxation, underrelaxation, or a weighted average as described in Ref. [2].

In order to provide some guide to the selection of acceleration parameters for the most rapid convergence of the iterative solution, the optimum values were determined in a number of representative cases by computer experimentation. The results of these studies are given in Ref. [2] for use in operation of the computer program. The use of the variable acceleration parameter field is advantageous in some cases, particularly with strong coordinate attraction or other situations where there is large variance of coordinate line spacing over the field. The calculation of the variable acceleration parameter field, however, is time consuming, and the use of a constant parameter is more efficient in many cases.

## 3. Computer Code and Applications

### A. *Computer Code*

The computer code is independent of the boundary shapes and numbers, these being simply input values to the program. The various boundary segments of a multiple body field may be arranged in any way desired on the rectangular boundary of the transformed plane by simply adjusting input values. Points may be distributed around the boundaries as desired, with concentrations of points on some portions of the contour and more widely spaced points on others.

The coordinate system has a natural tendency to expand its line spacing from the inner boundary to the outer boundary. More concentration of coordinate lines in the vicinity of specified contours or points can be achieved by adjusting input values in the coordinate system control feature of the program. The variable acceleration parameter field for the SOR iteration will be calculated by the program if desired, or a uniform parameter may be input.

The program provides for several automatic convergence aids that may be activated by input parameters if necessary in particular cases, all of which are handled internally: (1) a choice of several different types of initial guesses for the iteration, (2) gradual addition of coordinate system control, and (3) gradual movement of outer boundary out to its final position.

Both printed and file storage output are available, and an internal contour plot routine provides plotted curvilinear coordinate lines on either a Calcomp or Gould plotter, all these options being controlled by input. The size of the plot and the area of the field covered are both adjustable.

The standard program allows a maximum field size of 70 $\xi$ lines and 60 $\eta$ lines and requires a core size of 52,400 words. Error signals and instructions for modification are given if these limits are exceeded. Typical computer times are in the range of a few minutes on the UNIVAC 1106. The code is available in both the UNIVAC and the CDC forms (Ref. [2]).

A second program is also included in Ref. [2] that calculates the scale factors from the coordinates for use in the solution of partial differential equations on the coordinate system. These factors are output to file storage. Any set of partial differential equations may be solved on the curvilinear coordinate system by programming the transformed equations using these coordinate scale factors. This procedure is illustrated in Ref. [13].

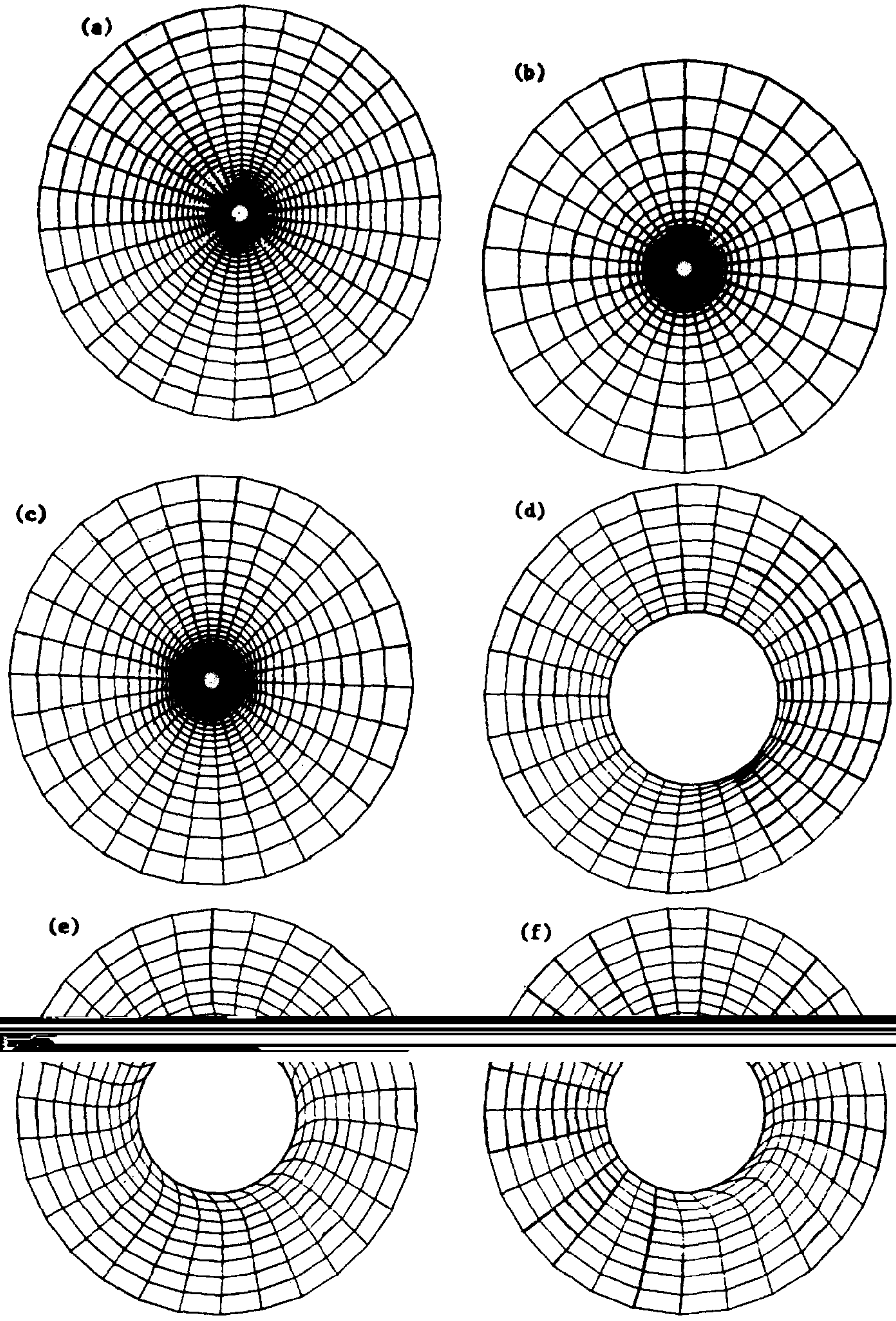


FIG. 13. Examples of coordinate system control.

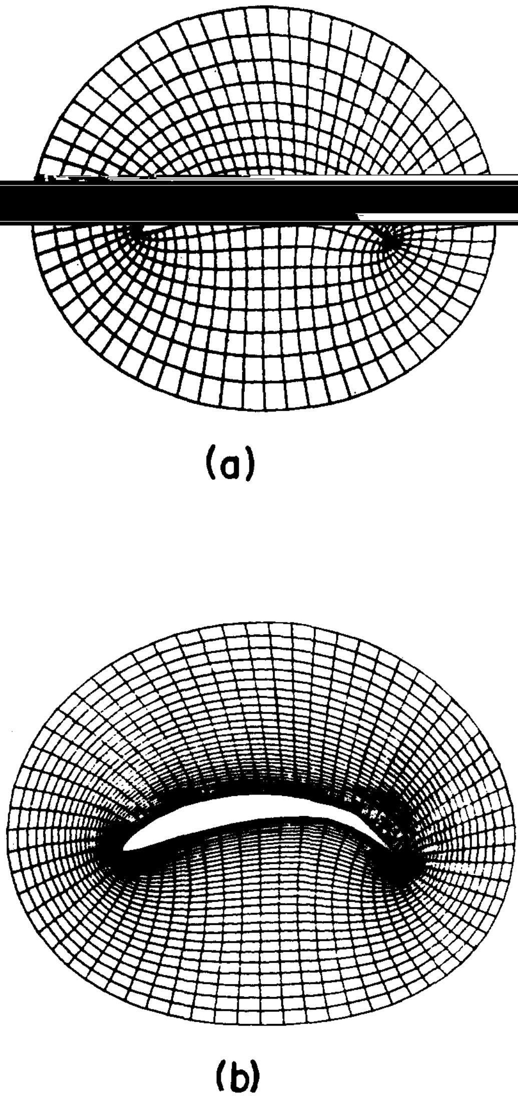


FIG. 14. Example of attraction into concave region.

B. *Coordinate System Control*

As discussed in Section 2, the curvilinear coordinate lines may be concentrated by attracting the lines to other lines or points in the field. The control of the coordinate system in this manner is illustrated in Figs. 13 and 14. Input parameters involved are given in Ref. [2]. In Fig. 13 the basic system generated by the Laplace equations (zero right-hand sides) is shown in (a). In (b) the $\eta$-lines have been attracted to the

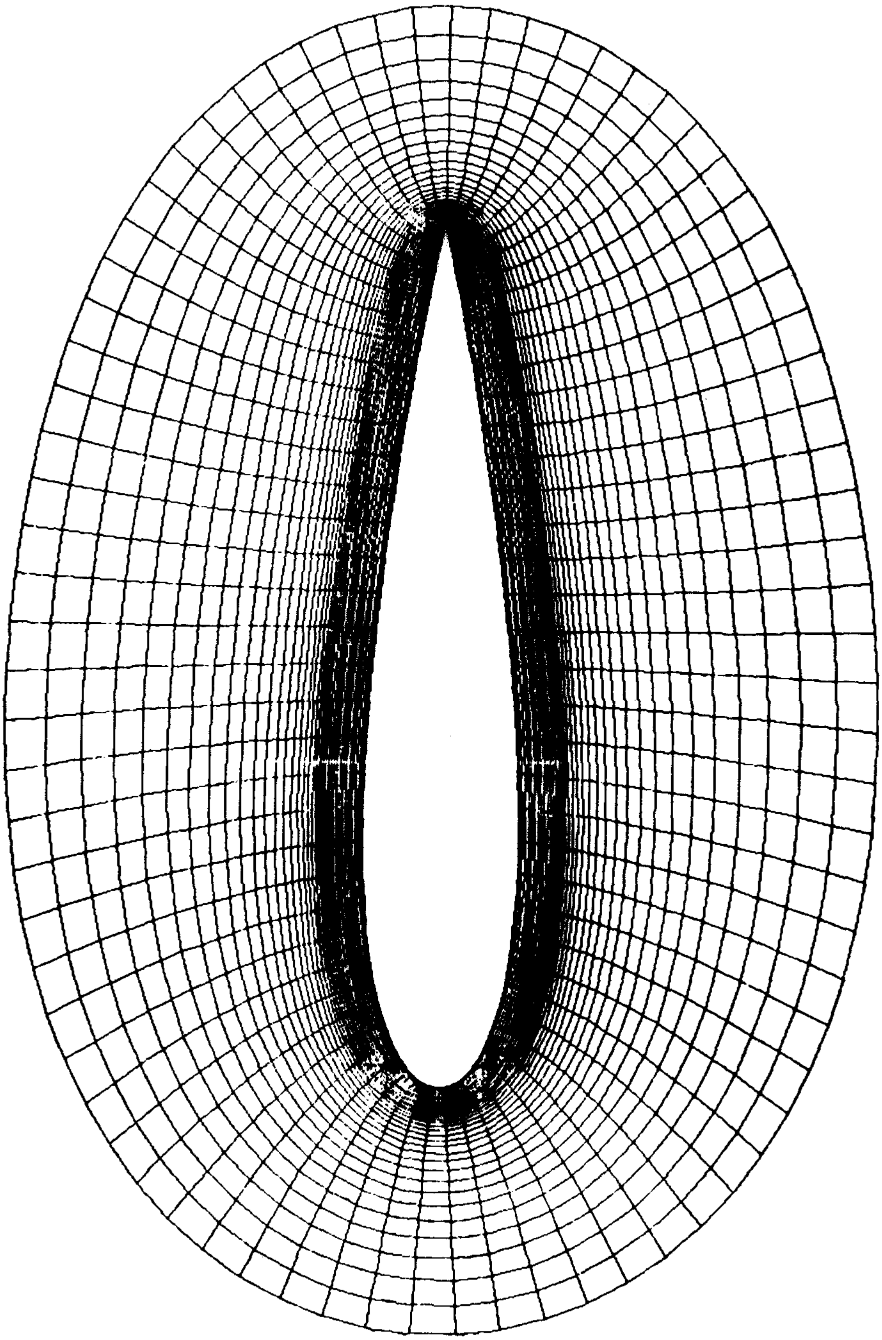

FIG. 15. Contracted coordinate system—NACA 0018 airfoil.

body. In (c) the attraction to the body has been made stronger on two sides, while in (d) the lines are more strongly attracted over a small portion of the body. In (e) and (f) the angle of intersection of the lines with the body has been controlled, over the entire body in (e) and over only a portion of the body in (f).

Figure 14 illustrates the use of control to pull the coordinate lines into a concave portion of the body contour, (a) being the result of the Laplace equations, and (b) having the lines attracted to the body and to the slope discontinuity on the lower surface.

## C. *Various Body Shapes*

Coordinate systems for a number of boundary shapes and configurations are given in Ref. [2], as well as in Refs. [11–15]. A few examples are shown herein in Figs. 14–18.

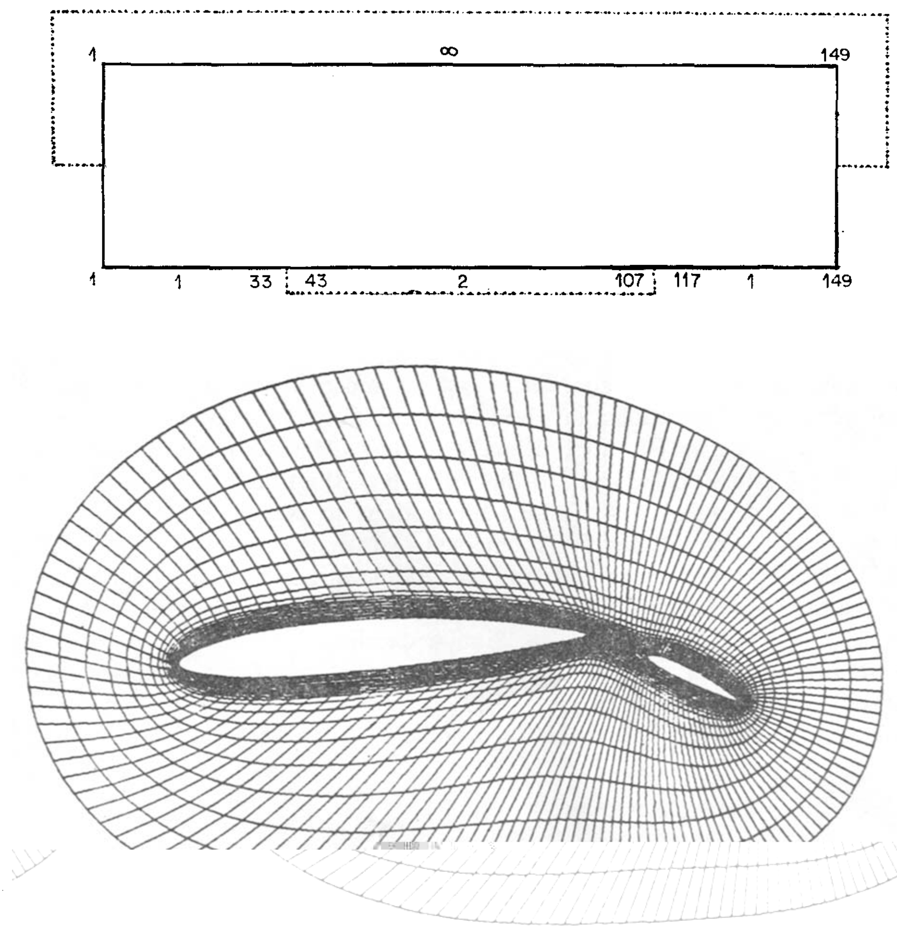


FIG. 16. Contracted coordinate system—Multiple airfoil.

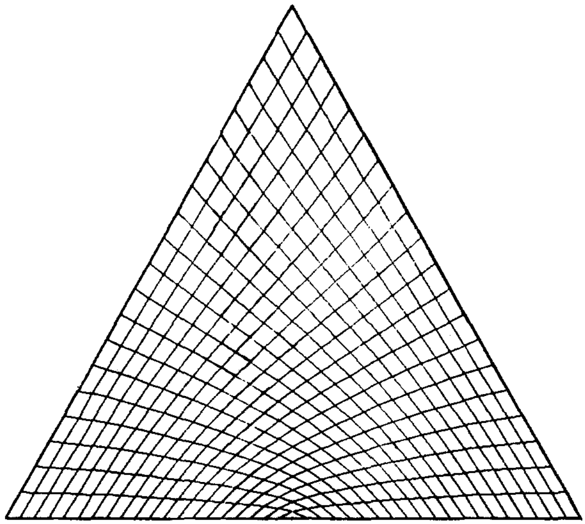

FIG. 17. Coordinate system—triangular simply-connected region.

Figure 16 shows a contracted coordinate system for a multiple airfoil, while Figs. 17 and 18 show simply connected regions.

D. *Application to Solutions of Partial Differential Equations*

A number of examples of coordinate systems and their use in the solution of the partial differential equations involved in fluid flow have been given in Refs. [1, 2, 11–15]. In particular, application is made to potential flow about multiple bodies (Fig. 19) in Ref. [13] and to the solution of the full incompressible Navier–Stokes equations in Ref. [14]. Initial applications to Navier–Stokes solutions for multiple airfoils (Fig. 20) and for submerged hydrofoils have also been reported in [12, 15]. Excellent agreement with the analytic solutions for lifting potential flow about Karman–Trefftz airfoils and multiple cylinders was reported in Ref. [13]. Examples are given in Ref. [2] also. Similar excellent agreement with the analytic solution has been obtained for the deflection contours for a simply supported uniformly loaded triangular flat plate (Fig. 21) using the coordinate system of Fig. 17, a problem involving solution of the biharmonic equation (Ref. [17]).

E. *Extension to Three Dimensions*

As noted above, the generation of curvilinear coordinate systems by solution of elliptic systems is not restricted to two dimensions. Initial extension to three dimensions has been made, and potential flow solutions for several bodies having a plane of symmetry have been obtained using the three dimensional coordinates (Ref. [18]).

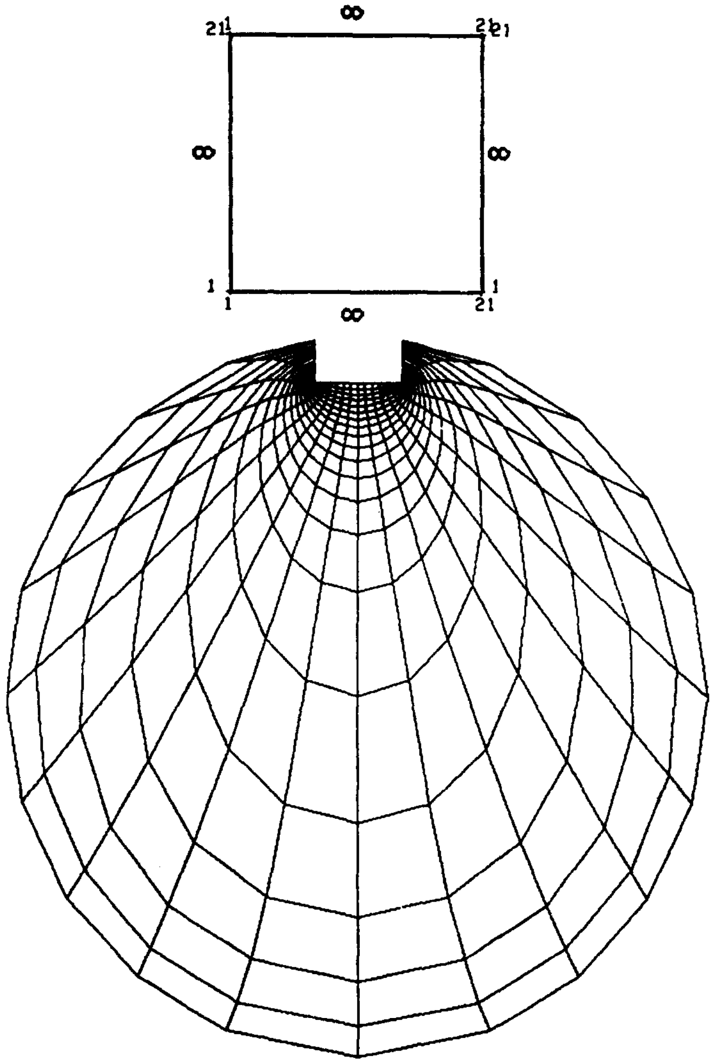


FIG. 18. Coordinate system—Key-seat shaft, simply-connected region.

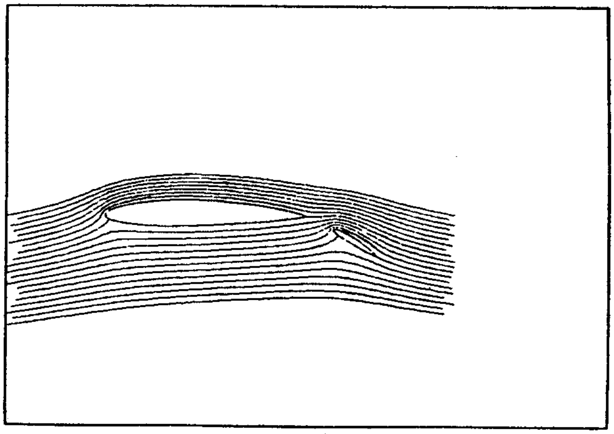

FIG. 19. Potential flow streamlines—Double airfoil.

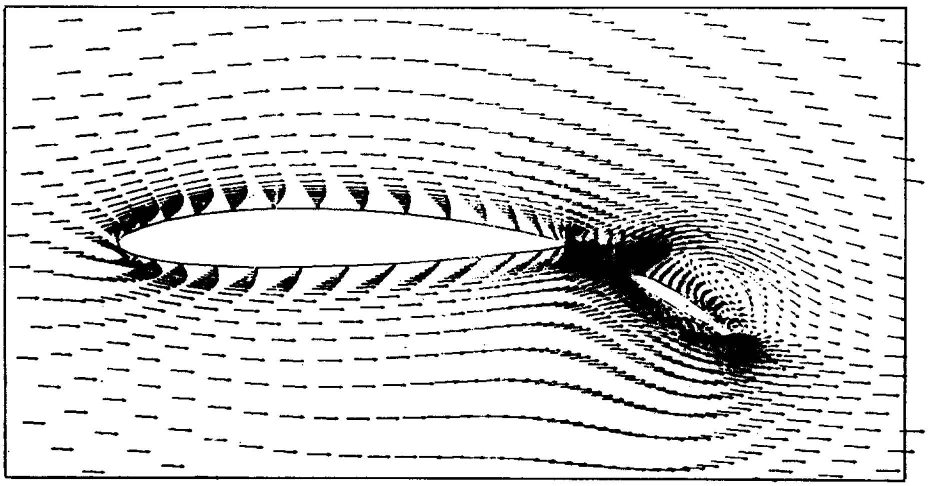

FIG. 20. Velocity vectors, viscous flow—Double airfoil, $R = 1000$.

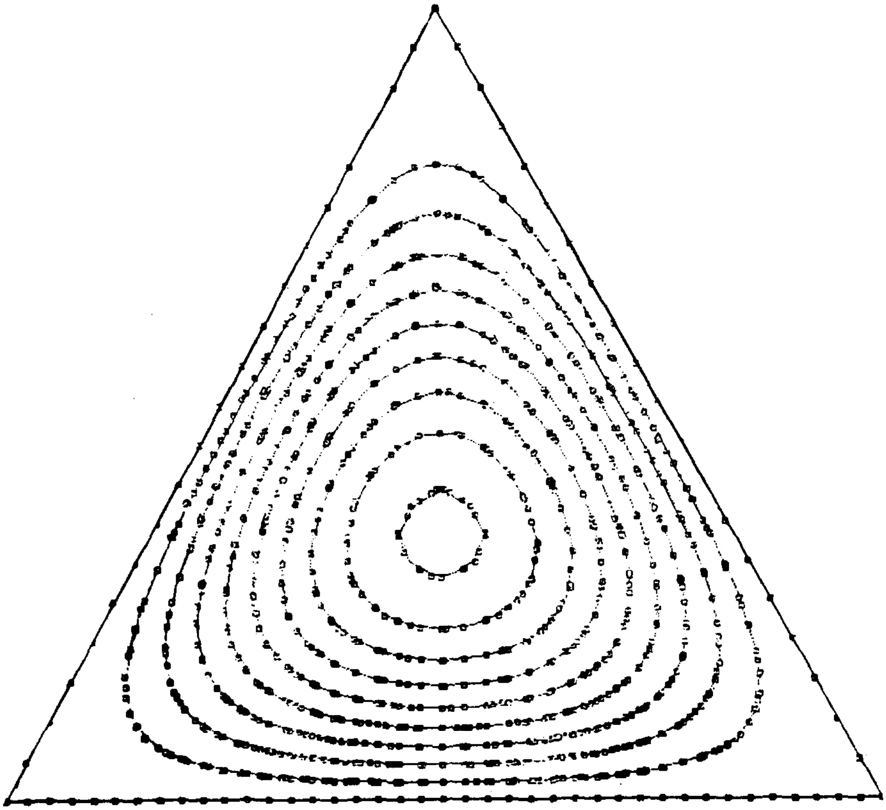

FIG. 21. Comparison of numerical and analytic solution for deflection contours for a simply-supported uniformly loaded Plate.

## 4. CONCLUSION

The boundary fitted curvilinear coordinate systems can eliminate, to a large degree, the problem of boundary shape in the numerical solution of partial differential equations. The efficiency of the procedure has been demonstrated by application to the solution of several partial differential systems. The boundary fitted coordinates may allow some problems that have been approachable only by finite element methods to now be treated by finite difference methods.

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